INSTRUMENTATION OF LOW FREQUENCY NOISE THROUGH WAVELETS

by
AMIT SAXENA

DEPARTMENT OF ELECTRICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

MARCH, 1995

INSTRUMENTATION OF LOW FREQUENCY NOISE THROUGH WAVELETS

A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of

MASTER OF TECHNOLOGY

by

AMIT SAXENA

to the

DEPARTMENT OF ELECTRICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

March 1995

Certificate

This is to certify that the work contained in this thesis, entitled *INSTRUMENTATION OF LOW FREQUENCY NOISE THROUGH WAVELETS* has been carried out by *AMIT SAXENA* under our supervision and that this work has not been submitted elsewhere for a degree

Dr R Sharan
Dept of EE,
IIT Kanpur

Dr Joseph John
Dept of EE
IIT Kanpur

March 1995

Dedicated to

My Parents

ACKNOWLEDGEMENTS

I express my deep sense of gratitude to my thesis supervisor, Dr R Sharan for giving me valuable guidance and encouragement during this work

I would like to express my gratitude to my co guide Dr Joseph John for allowing me to work on wavelets

I would like to thank all my M tech batchmates for the memorable moments that we all had together Also I would specially like to thank Sanjay Bhavesh and Pai for helping me out of difficulties during the preparation of the thesis report I will always remember and cherish the good times I had playing cricket with my inmates of Hall 4 I shall also remember the workers in our hall canteen (especially chotu) for providing me with tea and eats with an eversmiling attitude even at odd hours Lastly I must mention the members of G 5 (Bhavesh, Sanjay Varda Pai and Venu) for the endless hours of thulla we had together

My stay here was possible due to the constant moral support provided by my family members

March 29,1995 Amit Saxena

Abstract

Measurement of noise is important in engineering A method to measure power spectral density (P S D) of low frequency noise ($1/\mid f \mid^{\gamma}$) has been developed In this method "Constant Q" filtering of noise is done For 'constant Q" filtering purpose Wavelet transform is used Wavelet transform has perfect reconstruction and "Constant Q" properties All cases of noises (different γs) are studied and algorithm is developed to give power spectral density of any type of noise Perfect reconstruction property of Wavelet transform is used to synthesize $1/\mid f \mid^{1}$ noise from white noise

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Study of noise is of importance in science and engineering because it sets lower limits to the accuracy of any measurement In order to evalute these limits one must know the magnitude and sources of noise involved and must learn how to measure and to locate these Generally one uses two prototypes of noise (1) White noise and (2) flicker noise or coloured noise

An easy way to distinguish between these noises is to consider their autocorrelation function $R_{xx}(\tau)$ as given by Keshner [1] and shown in fig 1 1

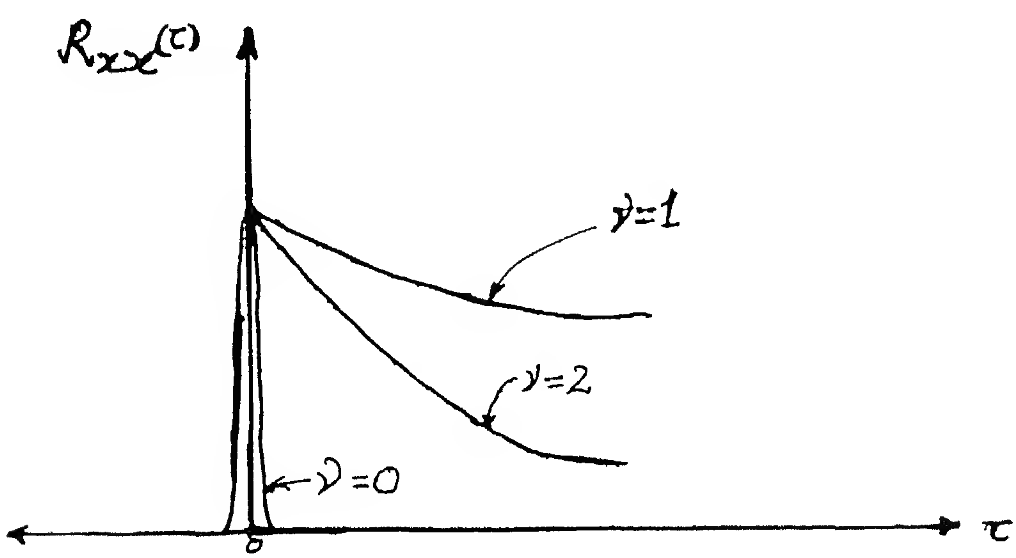


Figure 1 1 Autocorrelation function of 1/f noises

The white noise has the well known delta distributed autocorrelation function If

we represent the power spectral density of the noise as proprortional to $1/ \mid f \mid^{\gamma}$ then the white noise case corresponds to $\gamma = 0$ The case of $\gamma = 2$ is also well known because a variety of processes give rise to exponentially decaying autocorrelation function, examples of these processes are random telegraph signals, generation recombination noise in semiconductor etc The case of $\gamma = 1$ is called ' 1/f noise ' or 'flicker noise' and has the largest autocorrelation as evident from fig 1 1 Moreover in engineering practice, one finds that ' 1/f noise is ubiquitous and appears in several physical phenomena Some examples of these are [1],[2]

1 noise in the currents and voltages of electronic devices (e g field effect and bipolar transistors, schottky zener and tunnel diodes)

2 resistance fluctuations in metal film, semiconductor films and contacts ger manium filaments in carbon and aqueous solution thermocells and concen trations cells

3 burst errors in communication channels

4 physiological time series such as instantaneous heart rate records for healthy patients EEG variations under pleasing stimuli, and insulin uptake rate for diabetics

5 biological time series such as voltage across nerve and synthetic membranes

The study of flicker noise has been widely persued over almost six decades and even now several unresolved issues regarding it's modelling and it s affect on mea surement and communication are remaining The emphasis in the present work has been on the measurement issues and, particularly, an attempt has been made to exploit the recently developed wavelet theory [3] [8] for analysing and synthesizing 1/f noise as developed by Wornell [2] Our contribution in this work is in designing the filters and verifying the algorithms which relate to a theorem on 1/f noise as proved by Wornell [2] But before presenting this work, let us consider some simple schemes for the study of noise

Measurement of white noise can be done by constant bandwith filters as shown in figure 1 2(a) Here filters are of constant bandwidth, therefore power output of each filter will be the same But if noise is $1/\mid f \mid^{\gamma}$ and $\gamma = 1$, then it's measurement can be done by "Constant Q"or Constant relative bandwidth filters as shown in fig 1 2(b)

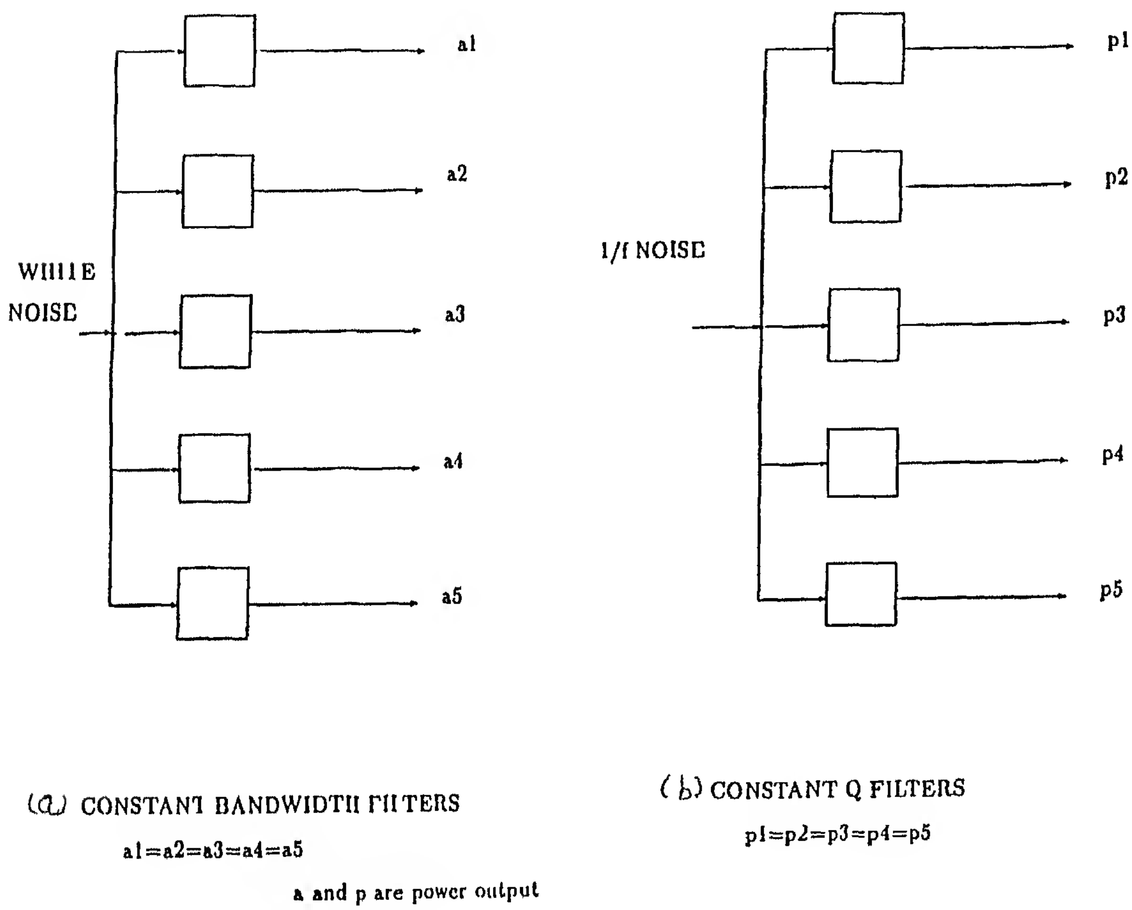


Figure 1 2 Constant Bandwidth And Constant Q Filters

Constant Q filters can be designed [14] through a number of existing methods other than that of wavelet based designing We can get Constant Q filter bank by designing each filter individually But this method will require a lot of computation Also with this method reconstruction will not be possible Whereas in Constant

Q filter designing through wavelets, only one sequence (which should fulfil certain conditions) is required Also this method ensures perfect reconstruction

In the present work "Constant Q" filters have been designed through wavelet theory These "Constant Q" filters are then used to give power spectral density of different noise sequences

To generate $1/f$ noise time series from white noise (in order to analyse it by Constant Q filters)we have used two methods

- Through digital filters designed using wavlet theory

- Through digital filter obtained from analog filter using matched Z transformation for analog to digital conversion Analog filter has a slope of -10db per decade

In first method a number of white noise sequences, having same power but different lengths are required, whereas in second case only one white noise sequence is required

Because in practice we can't have ideal white noise sequence, we have used Knuth's [11] white noise sequence generator whose sequential correlation (shown in plot in ch 5) is very near to that of white noise Because of this fact, noise after filtering can not be $1/f$ in the whole range 0 to π

1 1 organization

This thesis is divided into six chapters

- In chapter 2 wavelet theory is discussed In this chapter decomposition and reconstruction algorithm for any sequence is given Decomposition algorithm decomposes a signal in to different sequences in which each sequence corresponds to some frequency band of input sequence This therefore corresponds to Constant Q filtering of sequence

- In chapter 3, a theorem [2] is given, by which any type of noise can be synthesized from white noise In this chapter results are drawn from this theorem ,

to analyze any type of sequence through Constant Q filter bank

- In chapter 4 digital filter(s) design is given to generate $1/f$ noise Here both filter designing cases are discussed as stated earlier

- chapter 5 has results and relevant graphs

- In chapter 6 we have drawn conclusion/comparison based on above results

Chapter 2

Wavelet Theory

Wavelet theory provides a unified framework for a number of techniques which had been developed independently for various signal processing applications

In fact wavelet theory covers quite a large area It treats both the continuous and discrete-time cases It provides very general techniques that can be applied to many tasks in signal processing, and therefore has numerous potential applications In particular, the wavelet transform is of interest for the analysis of non stationary signals, because it provides an alternative to the classical STFT (short time Fourier transform) or gabor transform The basic difference between the two cases is in the choice of window[1]used In STFT a simple analysis window is used, whereas in wavelet transform short windows at higher frequencies and long windows at lower frequencies are used This is in the spirit of so called "Constant Q" or constant relative bandwidth frequency analysis

For some applications it is desirable to see the WT as a signal decomposition on to a set of basis functions In fact, basis functions called WAVELETS always underlie the wavelet analysis They are obtained from a single prototype wavelet by dilations and contractions as well as shifts The prototype wavelet can be thought of as a bandpass filter, and the constant Q property of the other bandpass filters (wavelets) follows because they are scaled versions of the prototype

[1]By window we mean a signal having a window like structure which has negligible amplitude outside certain range

There are several types of wavelet transforms, and depending upon the application, one may be preferred to the others For a continuous input signal, the time and scale parameters can be continuous, leading to a continuous Wavelet Transform(CWT) They may as well be discrete leading to a wavelet series expansion Finally the wavelet transform can be defined for discrete time signals leading to a discrete WT In the latter case it uses *Multiresolution signal processig technique* In this thesis we have used *Multiresolution signal processing technique* as a tool to get information about P S D of $1/|f|^{\gamma}$ noise, so we will emphasize on this method

2 1 A Review Of Multiresolution Analysis And Orthonormal Wavelet Bases

The idea of multiresolution analysis [3] is to write L^2–functions f as a limit of succesive approximations, each of which is a smoothed version of f, with more and more concentrated smoothing functions

More precisely, a multiresolution analysis consists of

(1) a family of closed embedded subspaces $V_m \subset L^2(R)^2$, $m \in Z^3$

$$\subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \tag{2 1}$$

such that (2)

$$\bigcap_{m \in Z} V_m = \{0\}, \overline{\bigcup_{m \in Z} V_m} = L^2(R) \tag{2 2}$$

and (3)

$$f \in V_m \leftrightarrow f(2\ ) \in V_{m-1}, \tag{2 3}$$

moreover, there is a $\phi \in V_0$ such that, for all $m \in Z$, the ϕ_{mn} constitute an unconditional basis for V_m, that is,(4)

$$\overline{V_m = linear\ span\{\phi_{mn}, n \in Z\}} \tag{2 4}$$

[2]Vector space of all measurable square integrable one dimensional functions

[3]Set of integers

and theie exit $0 < A \leq B < \propto$ such that, for all $C_{n n\in Z} \in l^2(Z)$,

$$A\sum_n | C_n |^2 \leq \| \sum_n C_n\phi_{mn} \|^2 \leq B\sum_n | C_n |^2 \qquad (2\,5)$$

Here $\phi_{mn} = 2^{-m/2}\phi(2^{-m}x - n)$ Let P_m denote the orthogonal projection onto V_m It is clear from 2 1, 2 2, that $lim_{m\to-\propto}P_m f = f$, for all $f \in L^2(R)$ The condition 2 3ensures that V_m correspond to different scales, while the translational invarience $f \in V_m \to f(\ - 2^m n) \in V_m$ for all $n \in Z$ is a consequence of 2 4 and 2 5 $\phi(x)$ is known as scaling function or father wavelet

Defining

$$C_{mn}(f) = < \phi_{mn}, f > \qquad (2\,6)$$

$$Then\ P_m f = \sum_n C_{mn}(f)\phi_{mn} \qquad (2\,7)$$

$P_m f$ is the approximated version of f in V_m vector space Now again if we ap proximate f at V_{m-1} then the information loss between two approximations can be obtained from projection theorem which is equal to the orthogonal projection onto orthogonal complement of V_m in V_{m-1} i e defining W_m as the orthogonal comple ment, in V_{m-1}, of V_m,

$$V_{m-1} = V_m \oplus W_m \quad , \ W_m \perp V_m \qquad (2\,8)$$

Equivalantely,

$$W_m = Q_m L^2(R) \quad with \quad Q_m = P_{m-1} - P_m \qquad (2\,9)$$

It follows immediately that all the W_m are scaled versions of W_0

$$f \in W_m \leftrightarrow f(2^m \ ) \in W_0 \qquad (2\,10)$$

and that the W_m are orthogonal spaces which sum to $L^2(R)$

$$L^2(R) = \bigoplus_{m\in Z} W_m \qquad (2\,11)$$

Wavelet theory ensures (eq 2 1 eq 2 5) that there exist $\psi \in W_0$ such that ψ_{mn} contitutes unconditional bases for W_m

This completes our analysis

2 2 SUMMARY *Multi resolution analysis(MRA)*

To start our analysis we choose ϕ satisfying 2 1 2 5, then we get $\psi(x)$ from $\phi(x)$ through following procedure Since $\phi \in V_0 \rightarrow \phi(2x-n) \in V_{-1}$, also since$V_0 \subset V_{-1}$ hence $\phi \in V_{-1}$ Thus there will exist C_n such that,

$$\phi(x) = \sum_n C_n \phi(2x-n) \tag{2 12}$$

Then we define

$$\psi(x) = \sum_n (-1)^n C_{n+1} \phi(2x+n) \tag{2 13}$$

ψ is called mother wavelet and ψ_{mn} are dilated(or contracted) and scaled versions of mother wavelet

2 3 Wavelet based decomposition and reconstruction algorithm

Let us assume that we have a sequence C_n^0 We want to decompose this signal in levels correspondoing to different frequency bands of input sequence

To do this we will use multiresolution analysis, i e we choose ϕ satisfying 2 1 2 5 After using this decomposition algirithm, sequences obtained will correspond to different frequency bands of input sequence In other words this algorithm will correspond to passing signal through Constant Q filters

2 3 1 decomposition algorithm

From C_n^0 we form a function f as follows

$f = \sum_n C_n^0 \phi_{0n}$, or

$f(x) = \sum_n C_n^0 \phi(x-n)$,

since V_0 is linear span of set $\{\phi_{0n}, n \in Z\}$ hence $f \in V_0$

Also from our construction $V_0 = V_1 \oplus W_1$, hence we can write,

$$f = P_1 f + Q_1 f \tag{2 14}$$

where $\phi_{1n} \rightarrow$ bases for V_1, and $\psi_{1n} \rightarrow$ bases for W_1 hence from 2 6 and 2 7 we can write,

$$P_1 f = \sum_n C_{1n}(f)\phi_{1n} \ , \ C_{1n}(f) = < \phi_{1n} \ f > \tag{2 15}$$

Where $P_1 f$ is approximation of f in V_1 vector space

Using similer technique we can write for $Q_1 f$

$$Q_1 f = \sum_n d_{1n}(f)\psi_{1n} \ , \ d_{1n}(f) = < \psi_{1n} \ f > \tag{2 16}$$

Where $Q_1 f$ is difference of information between f and it's approximated version $P_1 f$

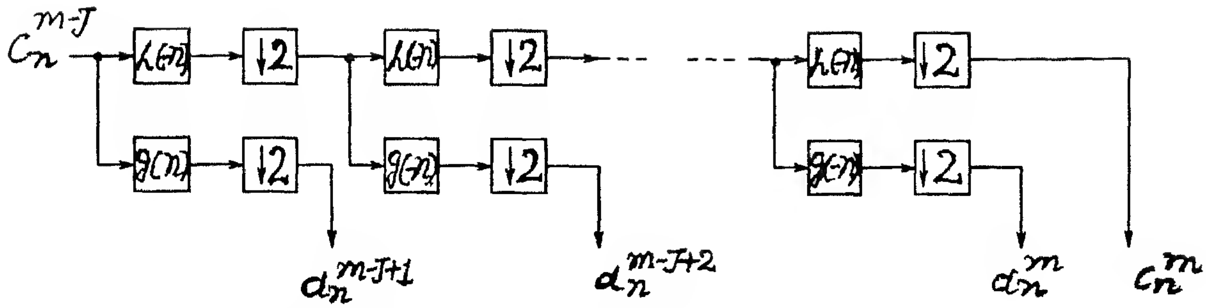
Writing c_n^1 for $C_{1n}(f)$ and d_n^1 for $d_{1n}(f)$

From 2 15

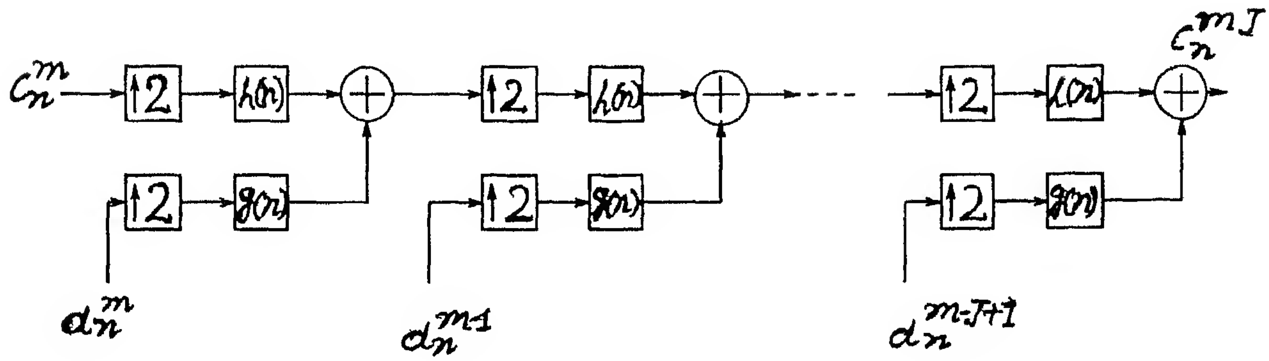
$$\begin{aligned}
c_n^1 &= < \phi_{1n}, f > \\
c_n^1 &= \int 2^{-1/2}\phi(2^{-1}x - n) f(x) dx \\
c_n^1 &= \int 2^{-1/2}\phi(2^{-1}x - n) \sum_k c_k^0 \phi_{0k} dx \quad from \quad above \\
c_n^1 &= \sum_k c_k^0 \int 2^{-1/2}\phi(2^{-1}x - n)\phi(x - k) dx \\
c_n^1 &= \sum_k 2^{-1/2} c_k^0 \int \phi(x/2)\phi(x - (k - 2n)) dx \\
c_n^1 &= \sum_k h(k - 2n) c_k^0
\end{aligned} \tag{2 17}$$

Where $h(n) = 2^{-1/2} \int \phi(x/2)\phi(x-n)dx$, is known as lowpass analysis filter Looking at eq 2 17 it is clear that c_n^1 from c_n^0 can be obtained by filtering it through $h(-n)$ and then subsampling it by 2 Similarly for d_n^1, from eq 2 16,

$$\begin{aligned}
d_n^1 &= < \psi_{1n} \ f > \\
d_n^1 &= < \psi_{1n} \sum_k c_k^0 \phi_{0k} > \quad from \quad above \\
d_n^1 &= \sum_k c_k^0 < \psi_{1n} \ \phi_{0k} > \\
d_n^1 &= \sum_k c_k^0 2^{-1/2} \int \psi(x/2 - n)\phi(x - k) dx \\
d_n^1 &= \sum_k c_k^0 2^{-1/2} \int \psi(x/2)\phi(x - (k - 2n)) dx \\
d_n^1 &= \sum_k c_k^0 g(k - 2n)
\end{aligned} \tag{2 18}$$



(a)DECOMPOSITION ALGORITHM



(b)RECONSTRUCTION ALGORITHM

Figure 2.1 Decomposition and reconstruction algorithm

where $g(n) = 2^{-1/2} \int \psi(x/2)\phi(x-n)dx$, is known as highpass analysis filter. Looking at eq. 2.18 and fig. 2.1(a) it is clear that d_n^1 from c_n^0 can be obtained by filtering it through $g(-n)$ and then subsampling it by 2. After getting c_n^1 from c_n^0 we can again apply the same analysis on c_n^1 to get c_n^2 and d_n^2, i.e. we can get lower resolution signals from higher resolution signals by repeating the analysis algorithm just described. This completes the analysis part.

2 3 2 Reconstruction algorithm

In reconstruction part (see figure 2 1) we have details(d^j_n ,j$\in [1,j]$) and one approximated signal(c^j_n) and we have to construct original signal c^0_n

We start with lowest resolution i e from j^{th} resolution, and construct a function α as shown below,

$$\alpha = \underbrace{\sum_n c^j_n \phi_{jn}}_{a} + \underbrace{\sum_n d^j_n \psi_{jn}}_{b} \qquad (2\ 19)$$

Clearly $a \in V_j$ and $b \in W_j$, hence $\alpha \in V_{j-1}$, and we can write $\alpha = P_{j-1}f$

$$\alpha = P_{j-1}f = \sum_n c^{j-1}_n \phi_{j-1n} \quad from \quad eq \quad 2\ 7 \qquad (2\ 20)$$

$$P_{j-1}f = \sum_n c^j_n \phi_{jn} + \sum_n d^j_n \psi_{jn} \qquad (2\ 21)$$

Also

$$\begin{aligned}
c^{j-1}_n &= < \phi_{j-1n}, P_{j-1}f > && (2\ 22)\\
c^{j-1}_n &= < \phi_{j-1n}, \sum_k c^{j-1}_k \phi_{j-1k} > from\ 2\ 20 \\
&= < \phi_{j-1n}, \sum_k c^j_k \phi_{jk} + \sum_k d^j_k \psi_{jk} > from\ 2\ 20 \\
&= \sum_k c^j_k < \phi_{j-1n}\ \phi_{jk} > + \sum_k d^j_k < \phi_{j-1n}, \psi_{jk} > \\
&= \sum_k c^j_k 2^{-1/2} \int \phi(x/2)\phi(x-(n-2k))dx + \sum_k d^j_k 2_{-1/2} \int \psi(x/2)\phi(x-(n-2k))dx \\
c^{j-1}_n &= \sum_k c^j_k h(n-2k) + \sum_k d^j_k g(n-2k) && (2\ 23)
\end{aligned}$$

i e we have got c^{j-1}_n from c^j_n and d^j_n

Thus recursively using above algorithm we can get c^0_n Reconstruction algorithm is shown in figure 2 1(b) in a schematic fashion

This completes decomposition and reconstruction algorithm which decomposes signal in to different frequency band signals and given these different frequency band signals, original signal can be constructed

To be more familiar with decomposition, reconstruction algorithms and detail and approximated signals we look at fig 2 1, and consider frequency spectra of these signals

Consider decomposition algorithm in which from c_n^0 (which will have frequency spectrum from 0 to π), first we get c_n^1 and d_n^1, where c_n^1 is obtained by filtering c_n^0 through h(n) and then subsampling it by 2 d_n^1 is obtained by filtering c_n^0 through g(n) and then subsampling it by 2 Here h(n) is half band low pass filter(pass band is 0 to $\pi/2$), and g(n) is half band high pass filter(pass band is $\pi/2$ to π) Hence c_n^1 will have information from 0 to $\pi/2$ of c_n^0 and d_n^1 will have information from $\pi/2$ to π of c_n^0 Again c_n^2 and d_n^2 are obtained from c_n^1 through similar procedure c_n^2 will have information from 0 to $\pi/2$ of c_n^1 and d_n^1 will have information from $\pi/2$ to π of c_n^1 In other words c_n^2 will have information from 0 to $\pi/4$ of c_n^0 and d_n^2 will have information from $\pi/4$ to $\pi/2$ of c_n^0

Hence after applying decomposition algorithm on a given signal (c_n^0),l times, we shall be having $(l+1)$ signals, in which l signals will be detail signals(from d_n^1 to d_n^l) and one will be approximated signal(c_n^l)

Here d_n^1 will correspond to $\pi/2$ to π of c_n^0

Here d_n^2 will correspond to $\pi/4$ to $\pi/2$ of c_n^0

Here d_n^3 will correspond to $\pi/8$ to $\pi/4$ of c_n^0

Here d_n^l will correspond to $\pi/2^{l-1}$ to $\pi/2^l$ of c_n^0

And c_n^l will correspond to 0 to $\pi/2^l$ of c_n^0

Hence we have divided original signal in to $(l+1)$ signals correspondig to different frequency bands of original signal Moreover what we have achieved through this decomposition is good frequency resolution at low frequencies and poor frequency resolution at higher frequencies (That is why it is called multiresolution analysis)

We can see two graphs given in chapter 5 right now in which White and colour noises are decomposed in to different d_n's and c_n by the filters designed in chapter 4 Looking at c_n^1 and d_n^1 (mentioned in these graphes as 1/2 resolution signals) one can very easi ly see that in case of white noise both c_n^1 and d_n^1 are rich in magnitude, showing that c_n^0 has good spectrum over the entire range 0 to π Whereas in colour noise case d_n^1 is very poor in magnitude as compared to c_n^1, showing that c_n^0 has very small spectrum in $\pi/2$ to π, which indeed is the fact

We have discussed types of WT in the begining of this chapter There was a wavelet series expansion represantation of signal We will discuss this representation in next section

2 4 Wavelet series expansion representation

An orthonormal wavelet transformation of a signal $f(x)$ is generally described in terms of synthesis/analysis equations [2]

$$f(x) = \sum_m \sum_n d_n^m \psi_{mn}(x) \tag{2 24}$$

where

$$\begin{aligned} d_n^m &= < \psi_{mn}, f > \\ &= \int f(x)\psi_{mn}(x)dx \end{aligned} \tag{2 25}$$

Considering synthesis equation i e eq 2 25 we can write

$$d_n^m = \int f(x) 2^{-m/2} \psi(2^{-m}x - n)dx \tag{2 26}$$

putting $2^{-m}x - n = 2^{-m}y$,

$$\begin{aligned} d_n^m &= \int f(y + 2^m n) 2^{-m/2} \psi(2^{-m}y)dy \\ &= \int f(y + 2^m n)\psi_{m0}(y)dy \\ d_n^m &= \{f(x) * \psi_{m0}(-x)\}_{|x=2^m n} \end{aligned} \tag{2 27}$$

This representation is known as *octave band filter band interpretation* Thus for each m, d_n can be obtained from *filter and sample operation* This representation is shown in figure 2 2

Considering analysis equation i e eq 2 24, it can be viewed as multirate modulation as depicted in figure 2 2

To understand how we get multirate modulation diagram as shown in fig 2 2 consider d_n^m, from eq 2 27, it is clear that for each m, d_n^m correspond to samples with consecutive samples 2^m seconds apart

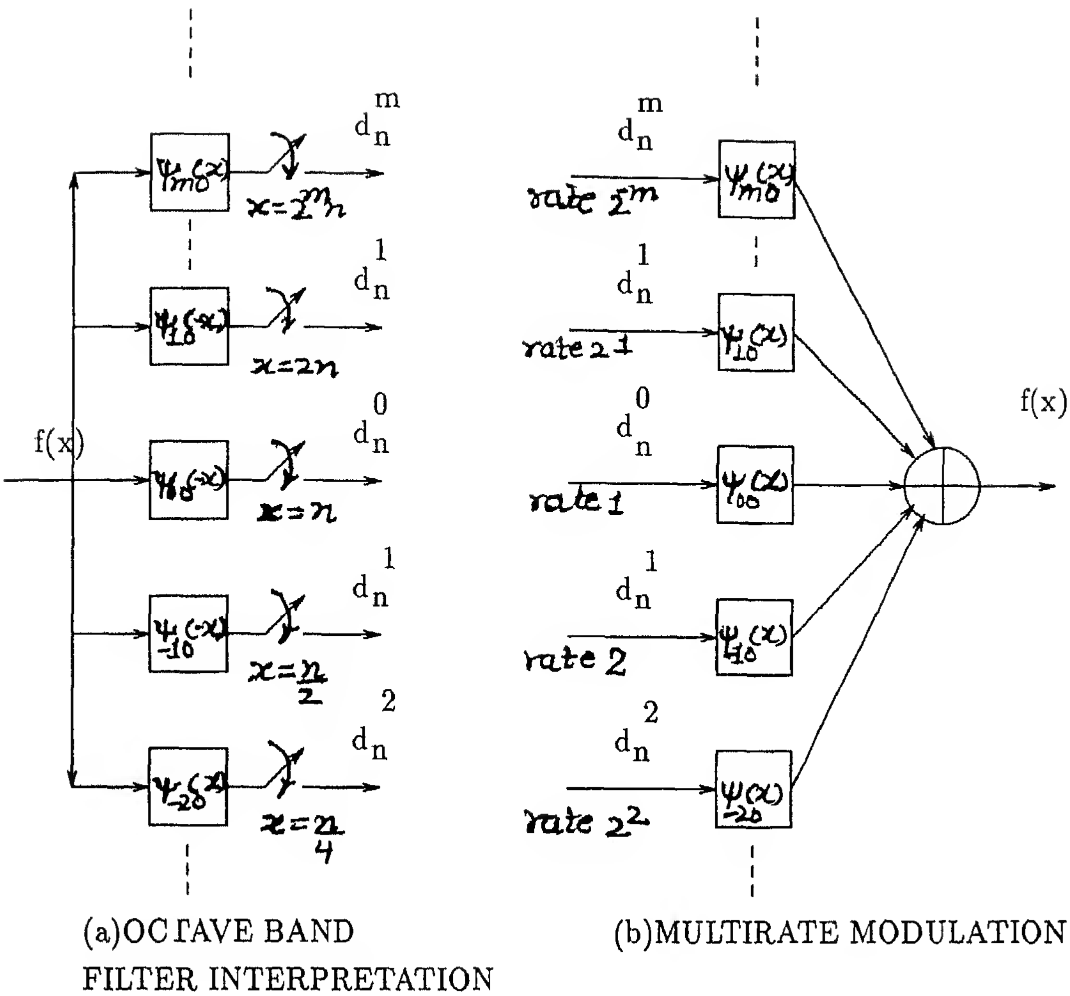


Figure 2 2 Octave band filter bank and multirate modulation interpretation

Hence d_n^m will have form as shown in fig 2 3 And d_n^m can be written as

$$d_n^m = \sum_{n_0} d_{n_0}^m \delta(n - n_0) \tag{2 28}$$

where $n = 2^{-m}x$ from octave band filter representation, therefore

$$\begin{aligned} d_n^m * \psi_{m0}(x) &= (\sum_{n_0} d_{n_0}^m \delta(2^{-m}x - n_0)) * \psi_{m0}(x) \\ &= \sum_{n_0} d_{n_0}^m (\delta(2^{-m}x - n_0) * \psi_{m0}(x)) \\ &= \sum_{n_0} d_{n_0}^m \psi_{m0}(x - 2^m n_0) \end{aligned}$$

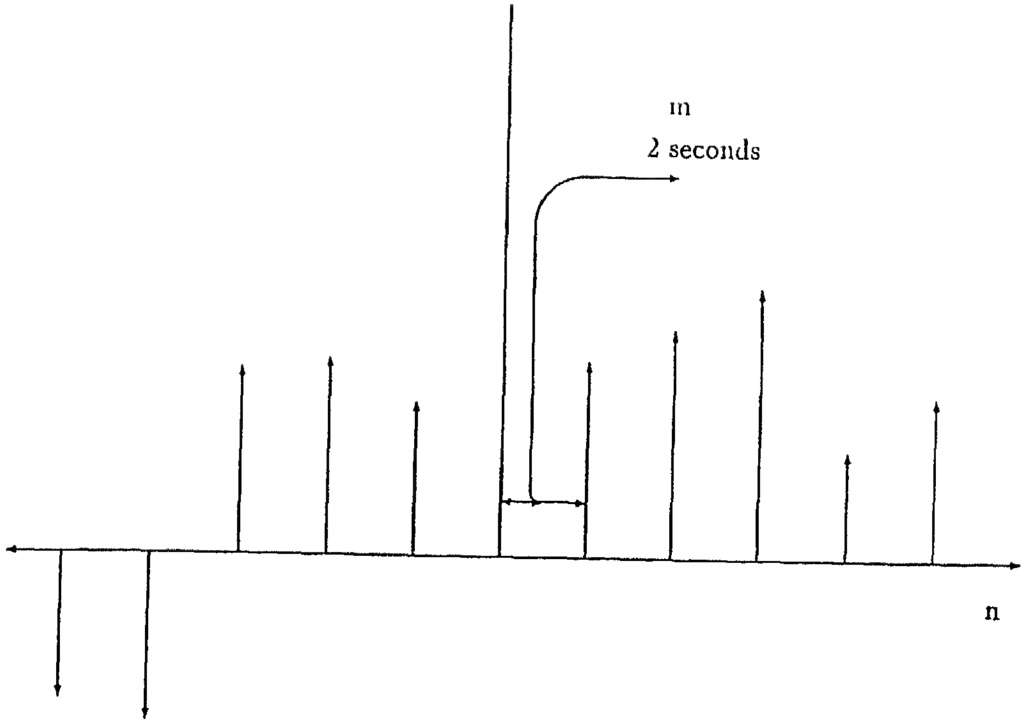


Figure 2 3 Sequence obtained from filter and sample operation

$$
\begin{aligned}
d_n^m * \psi_{m0}(x) &= \sum_n d_n^m \psi_{m0}(x - 2^m n) \\
&= \sum_n d_n^m 2^{-m/2} \psi(2^{-m}(x - 2^m n)) \\
&= \sum_n d_n^m 2^{-m/2} \psi(2^{-m} x - n) \\
d_n^m * \psi_{m0}(x) &= \sum_n d_n^m \psi_{mn}(x)
\end{aligned}
$$

or we can write

$$
\begin{aligned}
\sum_m (d_n^m * \psi_{m0}(x)) &= \sum_m \sum_n d_n^m \psi_{mn}(x) && (2\ 29) \\
&= f(x) && (2\ 30)
\end{aligned}
$$

i e synthesis equation can be viewed as multirate modulation as shown in figure 2 2

Chapter 3

Synthesis And Analysis of Low Frequency Noise

In chapter 2 we have discussed decomposition algorithm, which decomposes a signal in to different frequency bands This algorithm corresponds to Constant Q filtering of input signal

Now we will state a theorem [2] by which any type of low frequency noise can be synthesized through reconstruction algorithm of chapter 2 This theorem also helps in making the choice of wavelet basis that should be used to synthesize a particuler noise Initially we will motivate the structure of this theorem and then a rigereous statement will be made

In chapter 2 we discussed wavelet based series expansion Let us write the function $f(x)$ as follows

$$f(x) = \sum_{m} \sum_{n} d_n^m \psi_{mn}(x), \tag{3 1}$$

Now in order to represent a process, having power spectral density of the form $\sigma^2 / \mid f \mid^\gamma$, using equation 3 1 constraints have to be imposed on d_n^m and $\psi(x)$ These constraints are

1 d_n^m should be a collection of mutually uncorrelated zero mean random variables with variances [2]

$$var\{d_n^m\} = \sigma^2\{1/L_m\}^\gamma \tag{3 2}$$

Here L_m is length of sequence d_n^m Note that $L_m = 2^{-m}$ from fig 2 3 therefore

$\text{var}\{d_n^m\} = \sigma^2 2^{m\gamma}$

2 Orthonormal wavbelet basis $\psi(x)$ should be having at least $\gamma/2$ order regularity[1]

Now we will state the theorem [2]

THEOREM 3 1 *Consider any orthonormal wavelet basis with R^{th} order regularity for some $R \geq 1$ Then the random process constructed via the expansion,*

$f(x) = \sum_m \sum_n d_n^m \psi_{mn}(x)$

where d_n^m are a collection of mutullay uncorrelated zero mean random variables with variances,

$var\ \{d_n^m\} = \sigma^2 2^{m\gamma},$

for some parameter $2R > \gamma > 0$ has a time averaged spectrum

$S_f(\omega) = \sigma^2 \sum_m 2^{\gamma m} | \psi(2^m \omega) |^2$

that is nearly $1/f$ i e

$\frac{\sigma_L^2}{|\omega|^\gamma} \leq S_f(\omega) \leq \frac{\sigma_u^2}{|\omega|^\gamma}$

for some $0 < {\sigma_L}^2 \leq {\sigma_U}^2 < \infty$, and has octave spaced ripple i e for any integer K

$| \omega |^\gamma S_f(\omega) = | 2^K \omega |^\gamma S_f(2^K \omega)$

With the help of this theorem, an algorithm is developed in the present work to synthesize and analyze noises for different γs

Before proceeding towards next section we simplify eq 3 2 to get an equation for power(P_n) of sequence d_n^m as follows

$$var\{d_n^m\} = \sigma^2\{1/L_m\}^\gamma, \tag{3 3}$$

Note that

$$var\{d_n^m\} = \frac{1}{L_m} \sum_{n=1}^{L_m} \{d_n^m\}^2 \tag{3 4}$$

[1]see Appendix A

Hence power(P_m),

$$P_m = \sum_{n=1}^{L_m}\{d_n^m\}^2 = \sigma^2\{1/L_m\}^{\gamma-1} \qquad (3\ 5)$$

or we can wrıte,

$$P_m = \sigma^2 2^{-m(1-\gamma)} \qquad (3\ 6)$$

3 1 Synthesıs of $1/\mid f\mid^\gamma$ noıse

To synthesıze $1/\mid f\mid^\gamma$ noıse for gıven σ^2, γ, frequency range and output sequence length L_{out}, steps that should be followed are gıven ın fig 3 1 Frequency range should be of the form π to $\pi/2^k$, where $k \in Z$ Also L_{out} should be a multıple of 2

To understand synthesıs more clearly we consıder an example Suppose we want a sequence of length 1024 (It ıs a multıple of 2) wıth P S D $\sigma^2/\mid f\mid^\gamma$ over frequency range π to $\pi/8$ In other words we have specıfied the values of σ^2, γ, frequency range and output sequence length, and usıng above theorem we want to construct a sequence, (from whıte noıse sequences) havıng P S D very near to as desıred

For thıs example $k = 3$ (see fig 3 1) and hence

1 Three detaıl sıgnals are requıred

   (a) $\pi/2$ to π sıgnal

   (b) $\pi/4$ to $\pi/2$ sıgnal

   (c) $\pi/8$ to $\pi/4$ sıgnal

2 Length of $\pi/8$ to $\pi/4$ sıgnal ıs

   $L_{out}/2^k = 1024/8 = 128$

   Sınce $L_m = 2^{-m}$ therefore $m = -7$ for $\pi/8$ to $\pi/4$ sıgnal Hence thıs sıgnal wıll be represented as d_n^{-7}

   Sımılarly length of $\pi/4$ to $\pi/2$ (d_n^{-8}) sıgnal = 256

   And length of $\pi/8$ to $\pi/4$ (d_n^{-9}) sıgnal = 512

3 Power requıred of d_n^{-7} sıgnal = $P_{-7} = \sigma^2 2^{-7(1-\gamma)}$ from eq 3 6

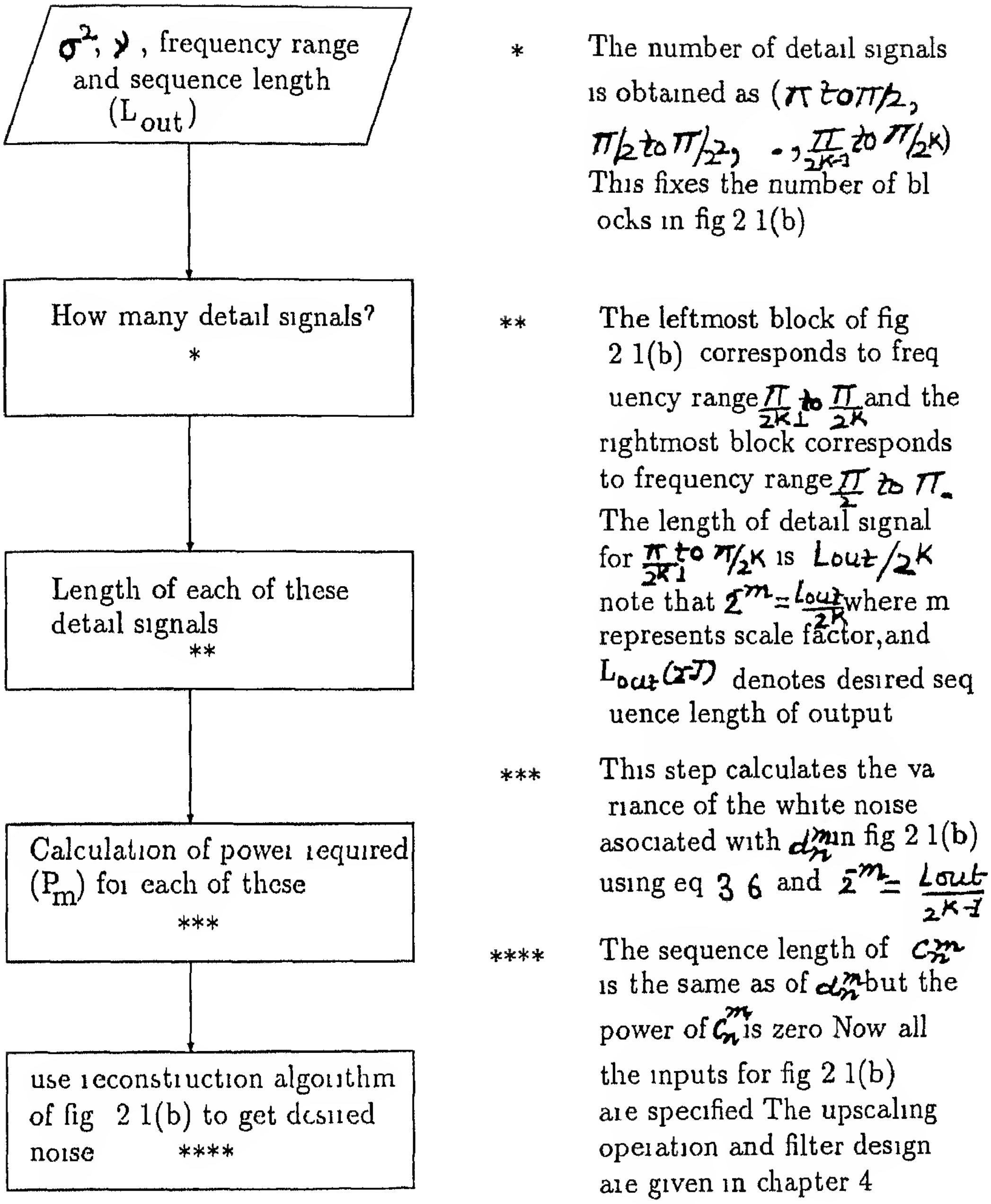


Figure 3 1 Steps for arranging the power and sequence length of input d_n^m and c_n^m in fig 2 1(b)

Power required of d_n^{-8} signal $= P_{-8} = \sigma^2 2^{-8(1-\gamma)}$

Power required of d_n^{-9} signal $= P_{-9} = \sigma^2 2^{-9(1-\gamma)}$

Hence by using d_n^{-7}, d_n^{-8} and d_n^{-9}, we can get c_n^{-10} through reconstruction algorithm which will have P S D very near to $\sigma^2/\mid f\mid^\gamma$ in the range $\pi/8$ to π

From above we have concluded following

1 To synthesize $1/\mid f\mid^\gamma$ noise for $1 > \gamma \geq 0$, lower resolution signals (higher values of m) must have lesser power than higher resolution signals (see eq 3 6) In other words to synthesize $1/\mid f\mid^\gamma$ noise for $\gamma < 1$ case we require zero mean uncorrelated sequences of different lengths As length of sequence increases(resolution increases) power of sequence must increase

2 To synthesize $1/\mid f\mid^\gamma$ noise for $\gamma = 1$ case, for all resolution signals, power required is same In other words to synthesize $1/\mid f\mid^1$ noise we require zero mean uncorrelated sequences of different lengths but of same power

3 To synthesize $1/\mid f\mid^\gamma$ noise for $1 < \gamma$, lower resolution signals (higher values of m) must have higher power than higher resolution signals (see eq 3 6) In other words to synthesize $1/\mid f\mid^\gamma$ noise for $\gamma > 1$ case, we require zero mean uncorrelated sequences of different lengths As length of sequence increases(resolution increases) power of sequence must decrease

3 2 Analysis of $1/\mid f\mid^\gamma$ noise

In analysis part we are given a sequence and σ^2 and γ of this sequence has to be found out This is done through "Constant Q" filter bank designed using wavelet transform (See fig 2 1(a)) Once we are given a sequence, and it has been passed through decomposition filters ("Constant Q" filters), we shall be having one approximated and other detail signals These signals will be of varying length with each decrease in resolution (increase in value of m), length will decrease to approximately half of its previous length Hence after passing given sequence through decomposition filters, we shall have P_m and L_m of each detail signal, and we want to know σ^2

and γ From eq 3 5 we have

$$P_m = \sigma^2 \{1/L_m\}^{\gamma-1} \tag{3 7}$$

or

$$\log_2 P_m = \log_2 \sigma^2 + (1-\gamma)\log_2 L_m \tag{3 8}$$

Looking at above equation, we can make following conclusions regarding γ of input sequence from the knowledge of P_m , L_m and the fact that as resolution decreases length of detail signal decreases These conclusions are

1 If power obtained in coarser detail signal is less than power obtained in finer detail signal, it is a $\gamma < 1$ case

2 If power obtained in each detail signal is same then its a $\gamma = 1$ case

3 If power obtained in coarser detail signal is more than power obtained in finer detail signal, it is a $\gamma > 1$ case

(** We could reach to the same conclusions as stated above in 1, 2, and 3 by looking at fig 3 2 In this figure power spectra of $\gamma < 1$ $\gamma = 1$ and $\gamma > 1$ noises are shown It is evident from this figure that power spectrum of $\gamma > 1$ noise converges faster than $\gamma = 1$ noise, whereas power spectrum of $\gamma < 1$ noise converges slower than $\gamma = 1$ noise

We are using "Constant Q" filters to analyse noise and if input noise has $\gamma = 1$ then power output of each of these filter will be the same But because of faster convergence of $\gamma > 1$ noise at higher frequencies (higher resolution) power output of higher resolution signal will we lesser than power output of lower resolution signals Similarly because of slower convergence of $\gamma < 1$ noise at higher frequencies power output of higher resolution signal will be higher than that of lower resolution signal **)

Therefore by plotting a graph between $\log L_m$ and $\log p_m$ for different detail signals obtained from "Constant Q filter bank, we can find σ^2 and γ In next section we have considered an example for analysis of $1/\mid f \mid^\gamma$ noise

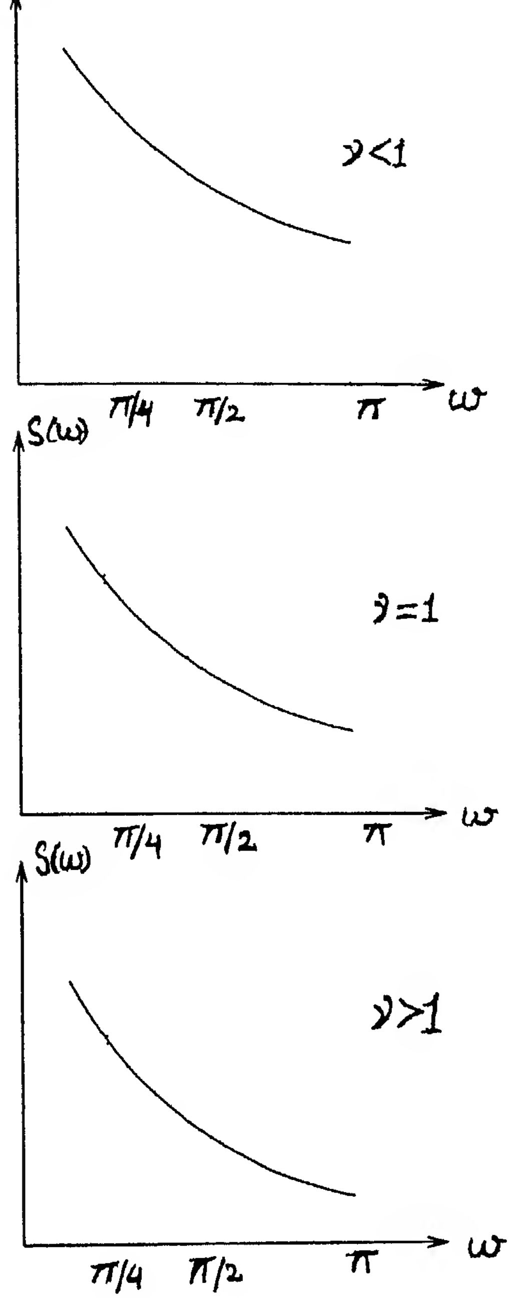


Figure 3 2 Power spectra of $1/\mid f\mid^{\gamma}$ noise

3 2 1 Example

Let us assume that we have a sequence of length 1024, and we want to know value of σ^2 and γ over certain frequency range Here frequency range will decide how many times decomposition algorithm has to be applied on the input signal If we want to know σ^2 and γ over frequency range (π to $\pi/2^k$), then decomposition algorithm (shown in fig 2 1(a)) will have k steps and there will be k detail signals

Let us assume that $k = 4$, hence first detail signal will be of length $\cong 512$ i e it can be represented as d_n^{-9}, similerly next coarser detail signal can be represented

as d_n^{-8} and so on Now we plot a graph between $\log_2 L_m$ and $\log_2 P_m$, let the graph obtained be as shown in fig 3 3

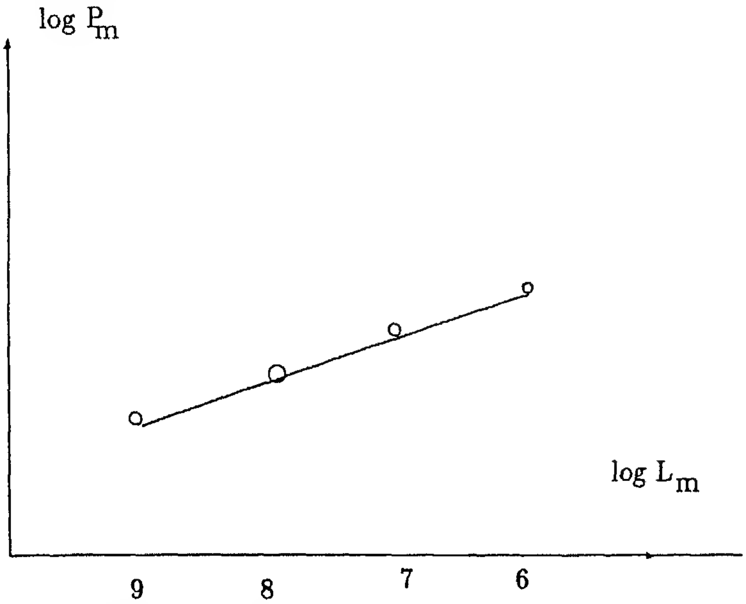


Figure 3 3 Plot between power and length for $\gamma > 1$ case

i e coarser detail signals are having more power than finer detail signals, it then is $\gamma > 1$ case and value of γ can be obtained from the slope of above curve, let slope be k, then

$\gamma - 1 = k$ and value of γ is obtained

If graph obtained is as shown in figure 3 4, then it's a $\gamma = 1$ case

However if graph obtained is as shown in figure 3 5, it is a $\gamma < 1$ case and value of γ is $\gamma = 1 - k$, where k is slope of the plot

After obtaining value of γ from the slope, σ^2 will be obtained from eq 3 8

Hence from Constant Q filter bank (realized by wavelet based decomposition algorithm) we can get P S D of any type of noise

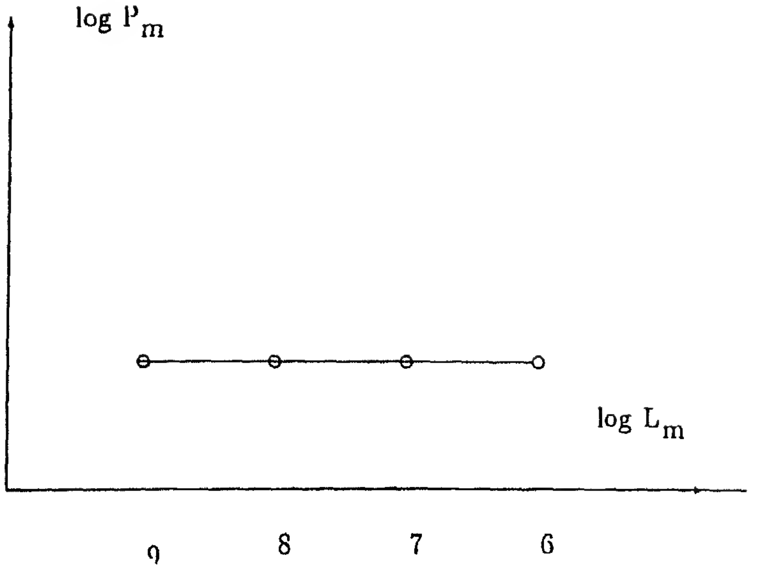


Figure 3 4 Plot between power and length for $\gamma = 1$ case

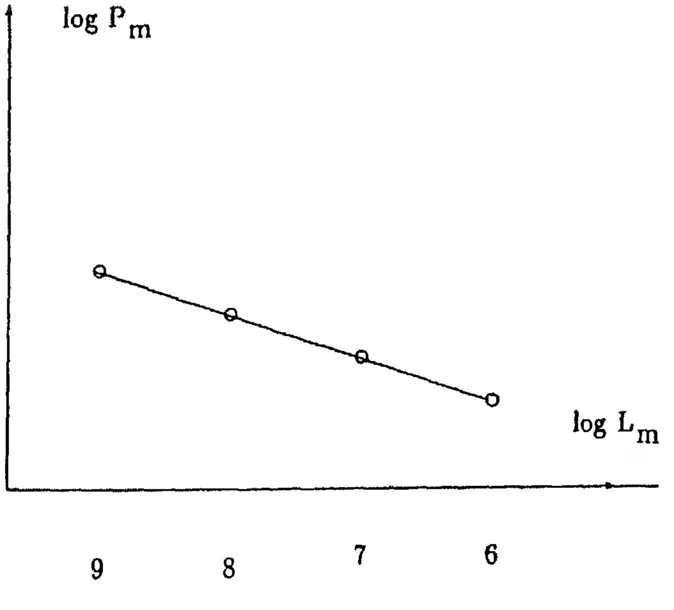


Figure 3 5 Plot between power and length for $\gamma < 1$ case

Chapter 4

Digital filter design

In [9],[10] $1/f$ like spectrum is obtained but that doesn't give an output time series, instead it first generates sequences through a recursive algorithm, then takes autocorrelation of each sequence and desired $1/f$ spectrum is obtained by taking FFT of each autocorrelation and then averaging these FFTs Since autocorrelation is not a linear operation, we can not get a sequence having $1/f$ like spectrum from this method

In chapter 3 we have discussed that $1/f$ noise can be synthesized from white noise sequences obeying certain power constraints We will discuss design aspects of these filters in next section

We will also generate $1/f$ noise sequence through digital filters from A/D trans formation using Matched Z transform, for analog to digital transformation with analog filter having a slope of 10db/decade Designing of this digital filter is dis cussed in section filter 2

4 1 filter 1

To realize decomposition ("Constant Q") and reconstruction filters, we have chosen Daubachies [3] 16 pt $h(n)$, reproduced in table 1, as low pass analysis filter This filter has very high regularity and obeys regularity constraint of theorem 3 1 for all practical cases of γs $(2 > \gamma \geq 0)$

n	h(n)	g(n)
0	054415842243	000117476784
1	312871590914	000675449406
2	675630736297	000391740373
3	585354683654	004870352993
4	015829105256	008746094047
5	284015542926	013981027917
6	000472484574	044088253931
7	128747426620	017369301002
8	017369301002	128747426620
9	044088253931	000472484574
10	013981027917	284015542926
11	008746094047	015829105256
12	004870352993	585354683654
13	000391740373	675630736297
14	000675449406	312871590914
15	000117476784	054415842243

Table 4 1 16 pt h(n) and g(n) used in decomposition and reconstruction algorithm [3]

Any other sequence can be chosen as $h(n)$ if it obeys conditions 1 to 3 of theorem given in appendix A and regularity condition of theorem 3 1

High pass analysis filter $g(n)$ is obtained from $h(n)$ by the relation [3],

$$g(n) = (-1)^n h(-n+1) \tag{4 1}$$

If $h(n)$ is of length 16 i e $n \in [0,15]$, $n \in Z$, then $g(n)$ will also be of length 16 with $n \in [-14,1]$, $n \in Z$ However $g(n)$ is shifted to right so that the shifted version $g'(n)$ will be zero for $n \notin [0,15]$, $n \in Z$

After choosing $h(n)$ and $g(n)$ as explained above, "Constant Q" filter bank is then realized by writting a computer program for cascade configuration shown in fig 2 1(a)

To realize reconstruction filter bank of fig 2 1(b) $h(-n)$ and $g(-n)$ are used, however both are again shifted to origin There is one difference between real algorithm written on computer for reconstruction filter bank and fig 2 1(b) This is as follows In computer program a backward shift of 15[1] is made after each summing block of fig 2 1(b) This backward shift is required because of delay caused by $h(-n)$ and $g(-n)$ filters

4 2 filter 2

$1/f$ can also be obtained from white noise by passing it through a filter having slope of 10db/decade slope This filter (analog) is realized by pole zero model with poles and zeros are as follows

pole1=3 hz zero1=8 5hz

pole2=30 hz zero1=95hz

pole3=300 hz

slope of this filter will be approx 10db/decade Digital filter is obtained by Matched Z transform for analog to digital transformation with sampling rate of 1000 hz (In Matched Z transform a zero is added at $Z = -1$ if number of zeros are less than number of poles) Difference equation of this filter is

$$\begin{aligned} y(n) &= x(n) - 499x(n-1) - 977x(n-2) + 526x(n-3) \\ &\quad +1\ 961y(n-1) - 1\ 087y(n-2) + 123y(n-3) \end{aligned} \tag{4 2}$$

A digital filter satisfying above difference equation is realized by writting a computer program

[1]Because filter length is 16 If filter length is l then shift should be $l-1$

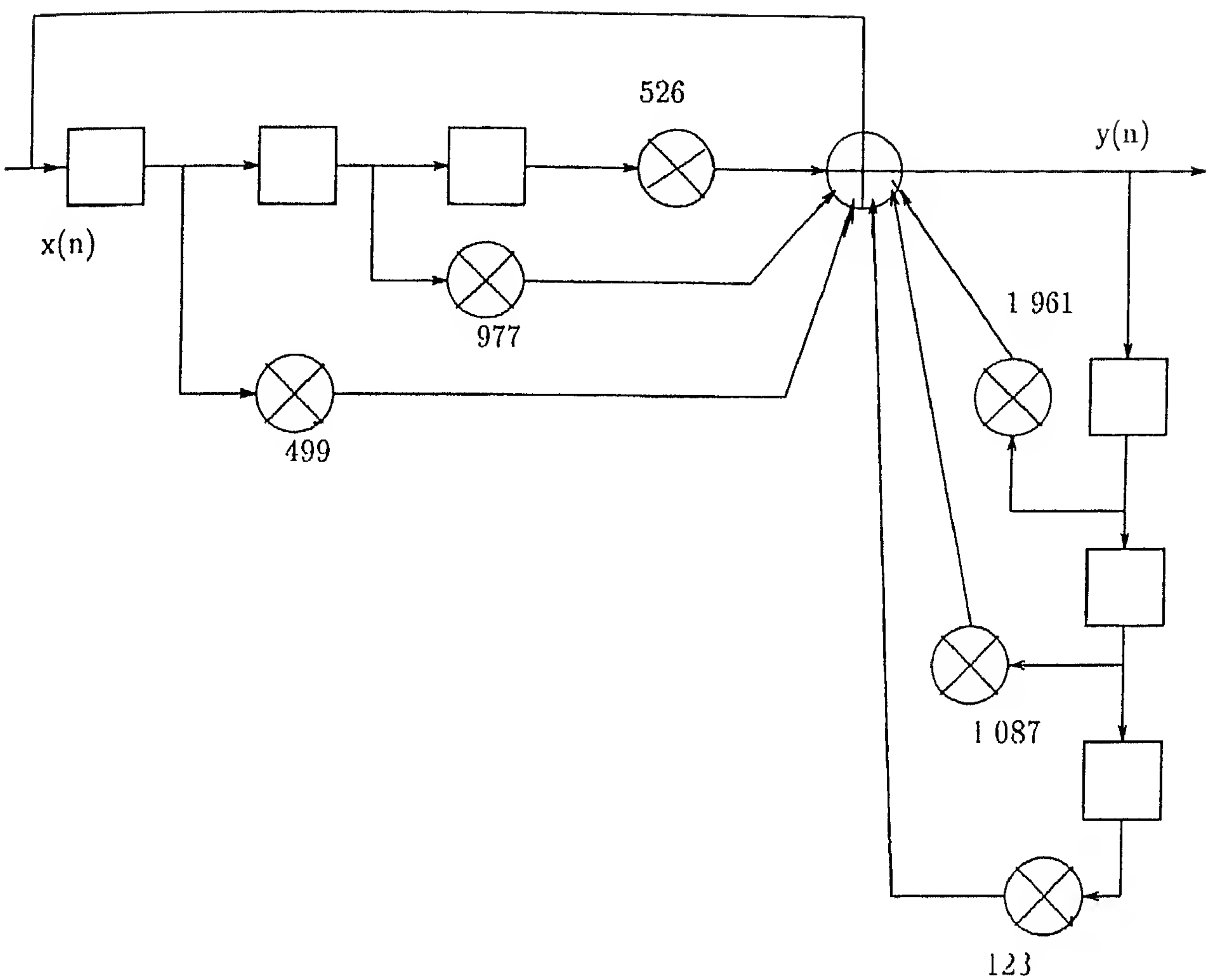


Figure 4 1 Digital filter through A/D conversion

Chapter 5

Results

In this chapter, the input and output waveforms and power spectral densities of the matched Z and wavelet based filters are given For the input and output of the matched Z filter the "detail" and "approximate" decomposition are also given Furthermore, the synthesis of $1/f$ noise through wavelet based filter bank has been presented

5 1 The input white noise sequence

An input white noise sequence has been used for both the filter 1 and filter 2 This sequence has been generated using Knuth s[11] algorithm and its autocorrelation and power spectral density (P S D) are shown in figs 5 1 and 5 2 respectively One notes that these are not ideal autocorrelation and ideal P S D for the sequence to qualify as white noise but are very good from the point of view of practical realization

5 2 The output $1/f$ noise of matched Z filter

1 The P S D of the output of the matched Z filter designed in chapter 4 to provide $1/f$ noise at the output, is shown in fig 5 3 Note that there is some deviation of this P S D from that of ideal $1/f$ noise and this can be attributed

to the nonideality of the input white noise as shown in fig 5 2

2 The time sequence (realization) of the input white noise and the output $1/f$ noise are shown in fig 5 4 A perusal of these plots exhibits the difference between the appearence of the white noise and $1/f$ noise as would be seen say, on an oscilloscope or a recorder

3 A noise sequence can be described in three different ways,

(a) by providing the realization, (b) by providing autocorrelation and P S D or (c) by providing the "detail and "approximate decomposition through wavelet analysis We have carried out the 'detail" and "approximate' decom position of all the waveforms studied in this thesis

The white noise detail" and "approximate" decomposition is shown in fig 5 5 to exhibit the features of multiresolution analysis based on wavelets Four blocks (i e four levels of resolution) of fig 2 1(a) have been used and the 'detail and 'approximate" output of each block is shown in fig 5 5 The 1/2 'detail' signal corresponds to the output of $(\pi - \pi/2)$ filter, and the 1/2 "approximate signal corresponds to the output of $(0 - \pi/2)$ filter, these are shown in fig 5 5(a) by red and blue respectively The 1/4 "detail" corresponds to $(\pi/2 - \pi/4)$ filter output and 1/4 "approximate' to $(0 \quad - \pi/4)$ filter output, and so on We will return to the considerations of this figure after presenting the 'detail' and approximate decomposition of $1/f$ noise for comparison

The "detail and approximate' decomposition of $1/f$ noise (obtained from the matched Z filter) are shown in fig 5 6 It is interesting to compare the 1/2 "approximate" and 1/2 detail' decomposition of the white and $1/f$ noise as given in fig 5 5(a) and 5 6(a) respectively One notes that the 1/2 detail" of $1/f$ noise is very much smaller in magnitude than the 1/2 'approximate signal One also notes that whereas the waveform ratio' of the 'detail" to the approximate' signals of white noise are the same for all the resolutions, this is not the case for $1/f$ noise For $1/f$ noise the 'detail" is comparatively small at 1/2 resolution and the 'waveform ratio" of the 'detail to the "approximate" signal incrases with decreasing (1/4 $\rightarrow$

1/8, $\rightarrow$ 1/16) resolution The lower resolution corresponds to lower frequency range in P S D , hence fig 5 6 shows that the "detail" signal increases at lower frequencies

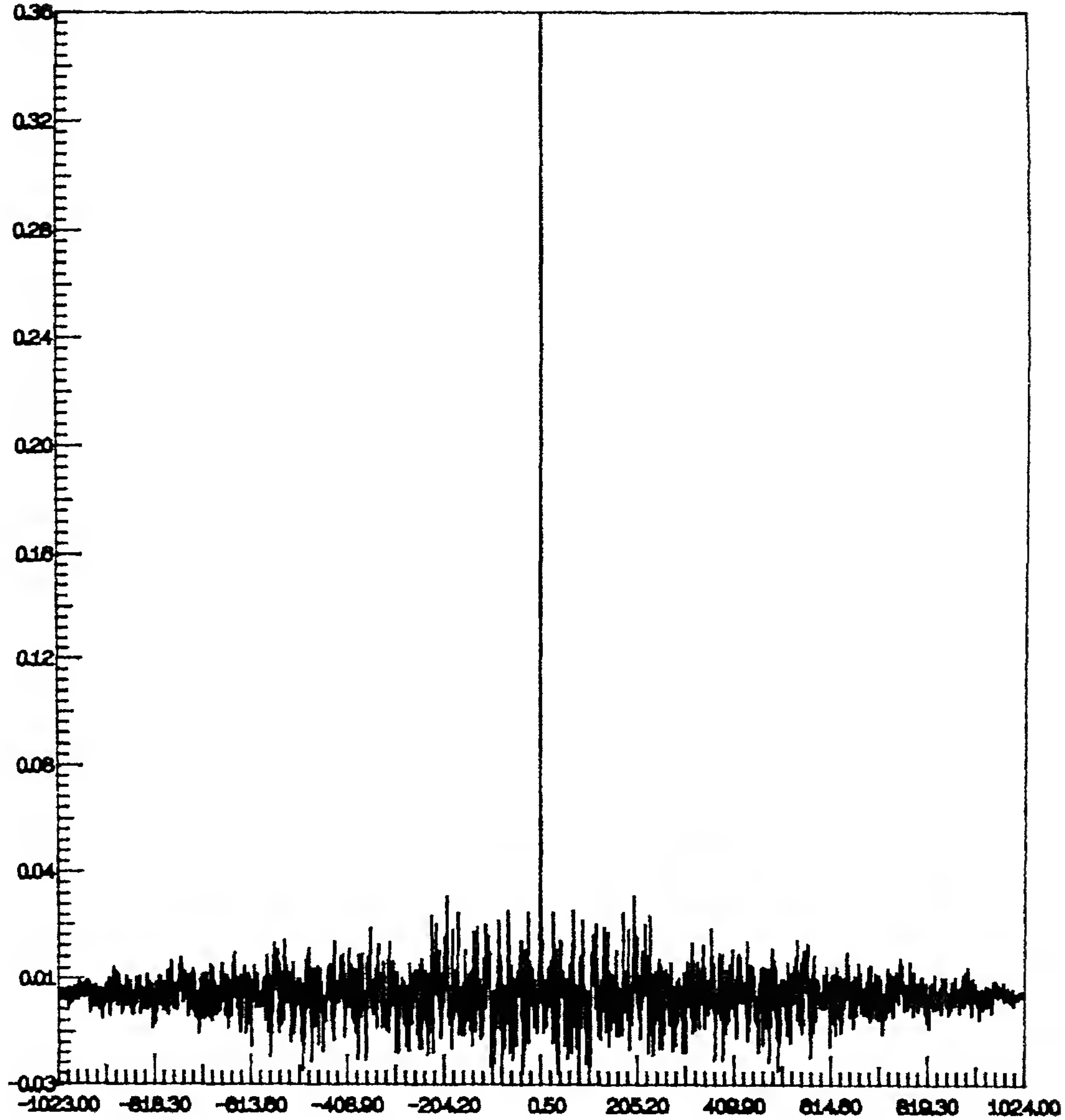
The output $1/f$ noise of the wavelet based filter can also be analyzed in a similar fashion, hence is not presented here

5 3 Synthesis of $1/f$ noise through wavelets

In the previous section the detail and "approximate" analysis of $1/f$ noise obtained as output of the matched z filter has been given Now we obtain $1/f$ noise as the output of a wavelet based filter bank (filter 1) shown in fig 2 1(b) The details of this filter are given in chapter 4

The input white noise to this filter is the same as described earlier and is shown in fig 5 7(a) for convenience and comparison The output realization of $1/f$ noise through this filter is shown in fig 5 8

To conclude, we emphasize that whereas the wavelet based filters have the capability of both analyzing and synthesizing a required noise waveform, the matched Z filter have only the capability of synthesizing a noise waveform However, the wavelet based filter for synthesis requires the input white noise sequences of different lengths but of same power, whereas the matched Z filter requires only one white noise sequence

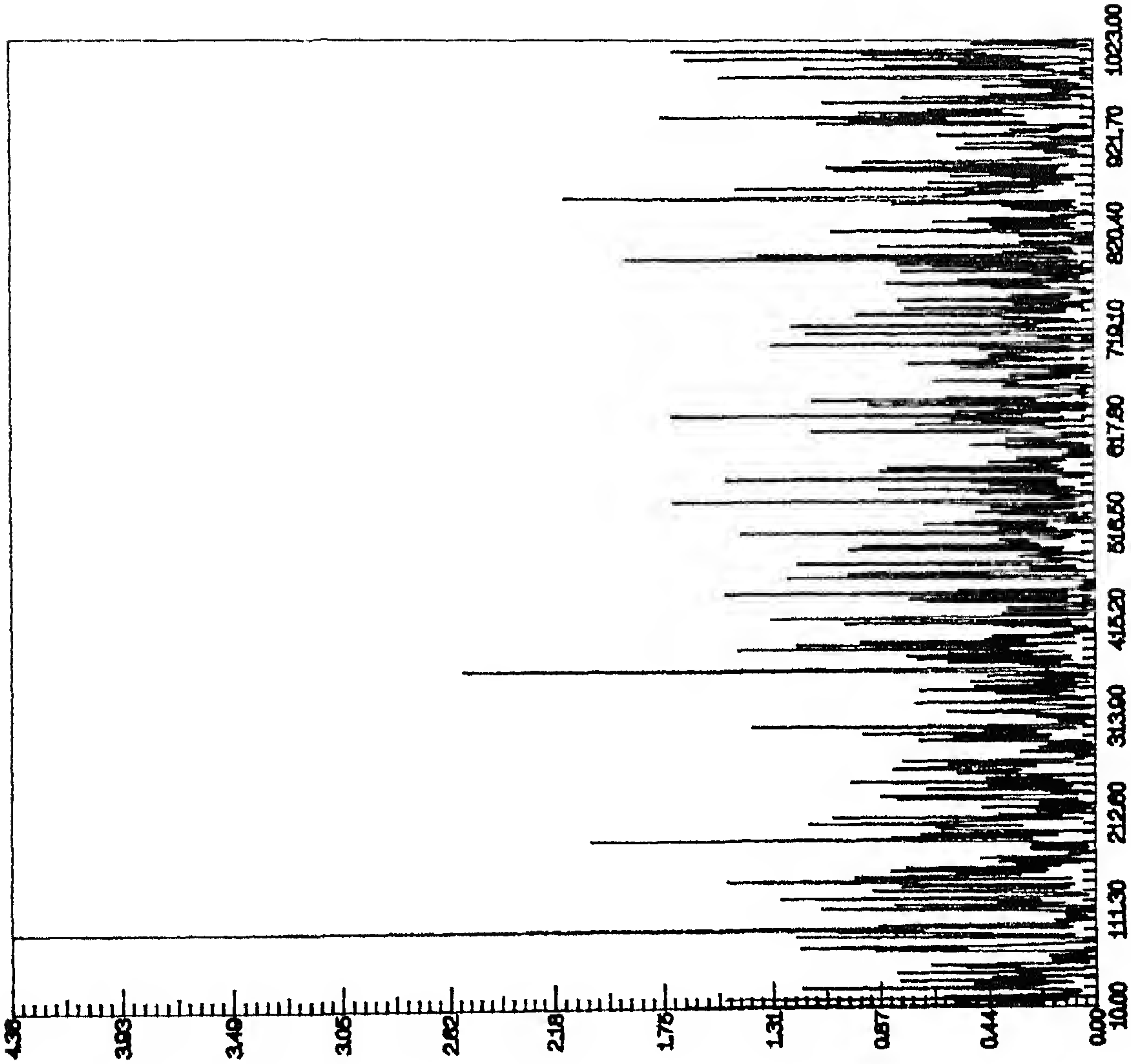


AUTOCORRELATION OF WHITE NOISE SEQUENCE

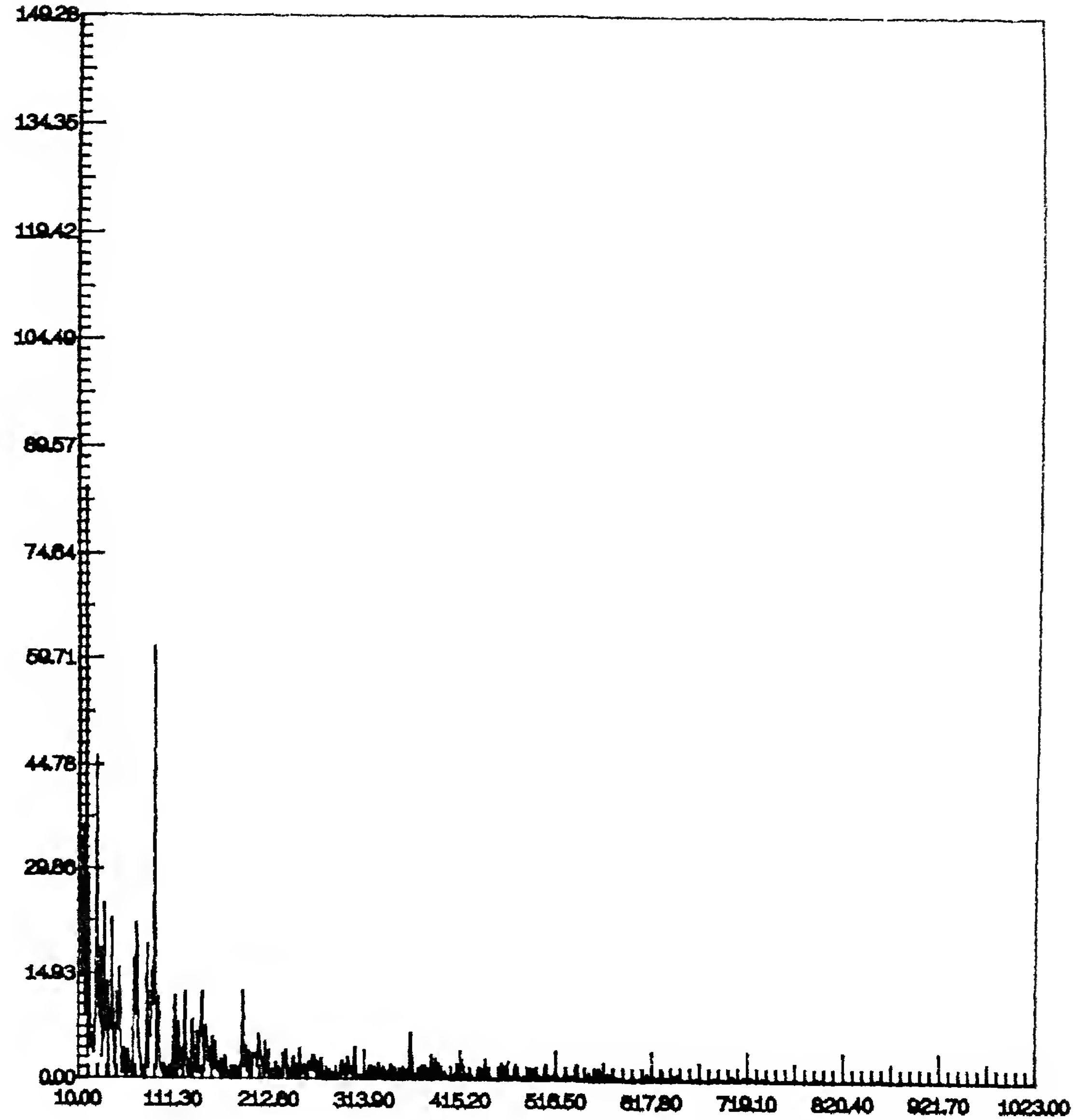
UsPlot

* asax * cc-iit k   prithvi * 03 13/95 09·42 55 *

UsPlot



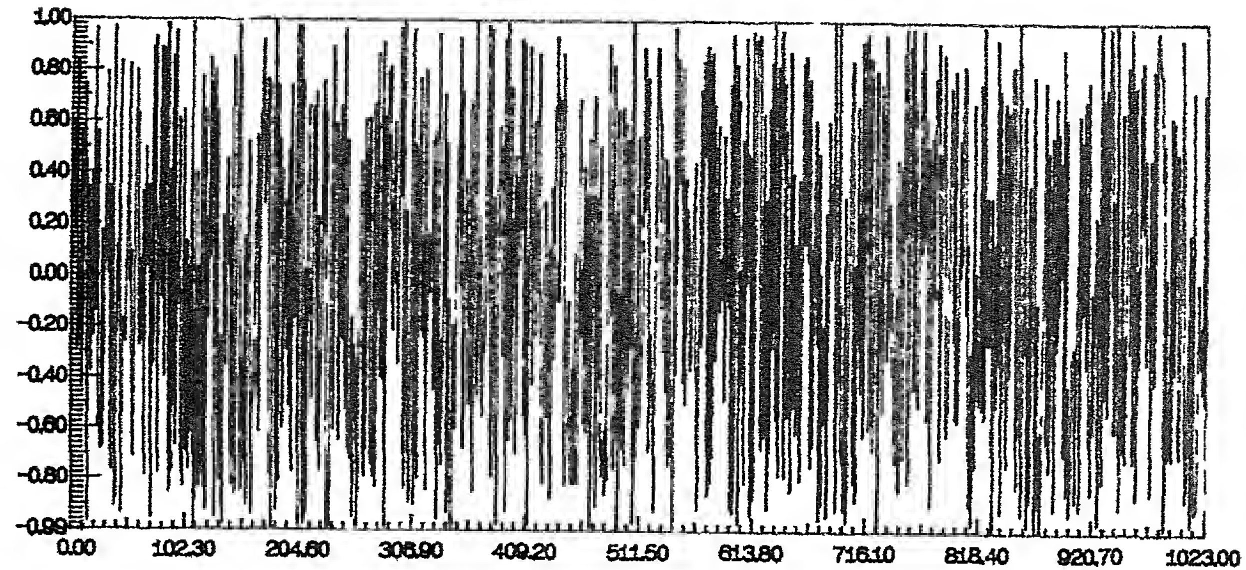
PSD OF WHITE NOISE SEQUENCE



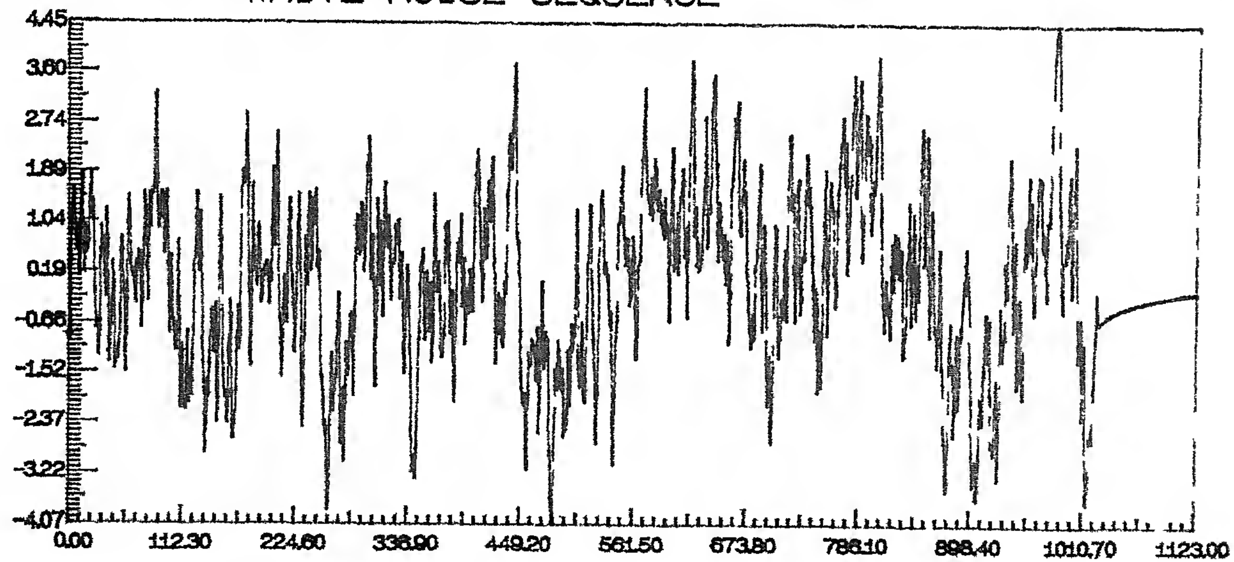
COLOURED NOISE OBTAINED THROUGH FILTER-2

UsPlot

amax # cc-iit/k   prithvi # 03/13/95 09:47:40



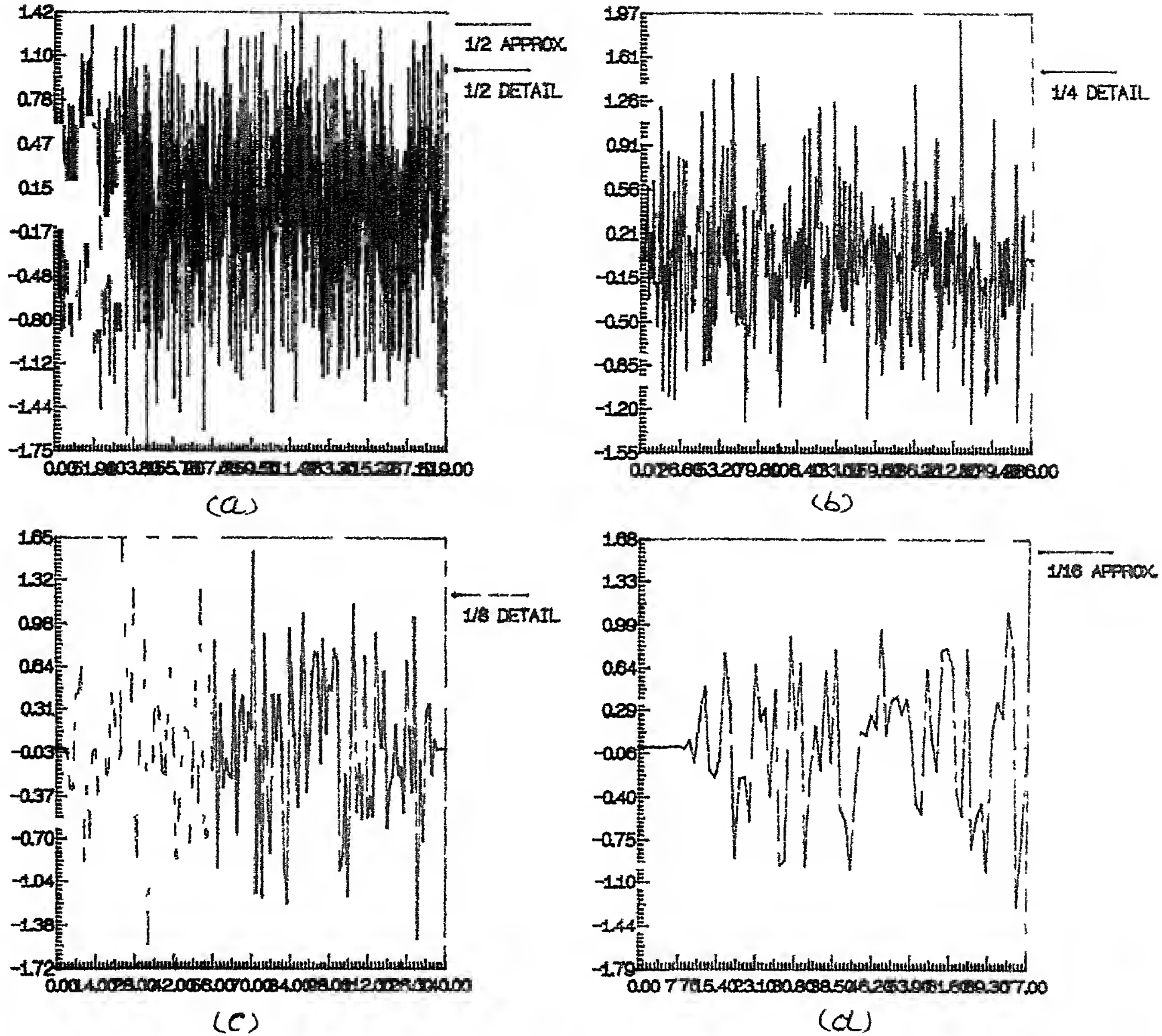
WHITE NOISE SEQUENCE



COLOURED NOISE OBTAINED THROUGH FILTER-2

UsPlot

* ngpai * cc-iit/k tejas * 03/13/95 09:39:45 *

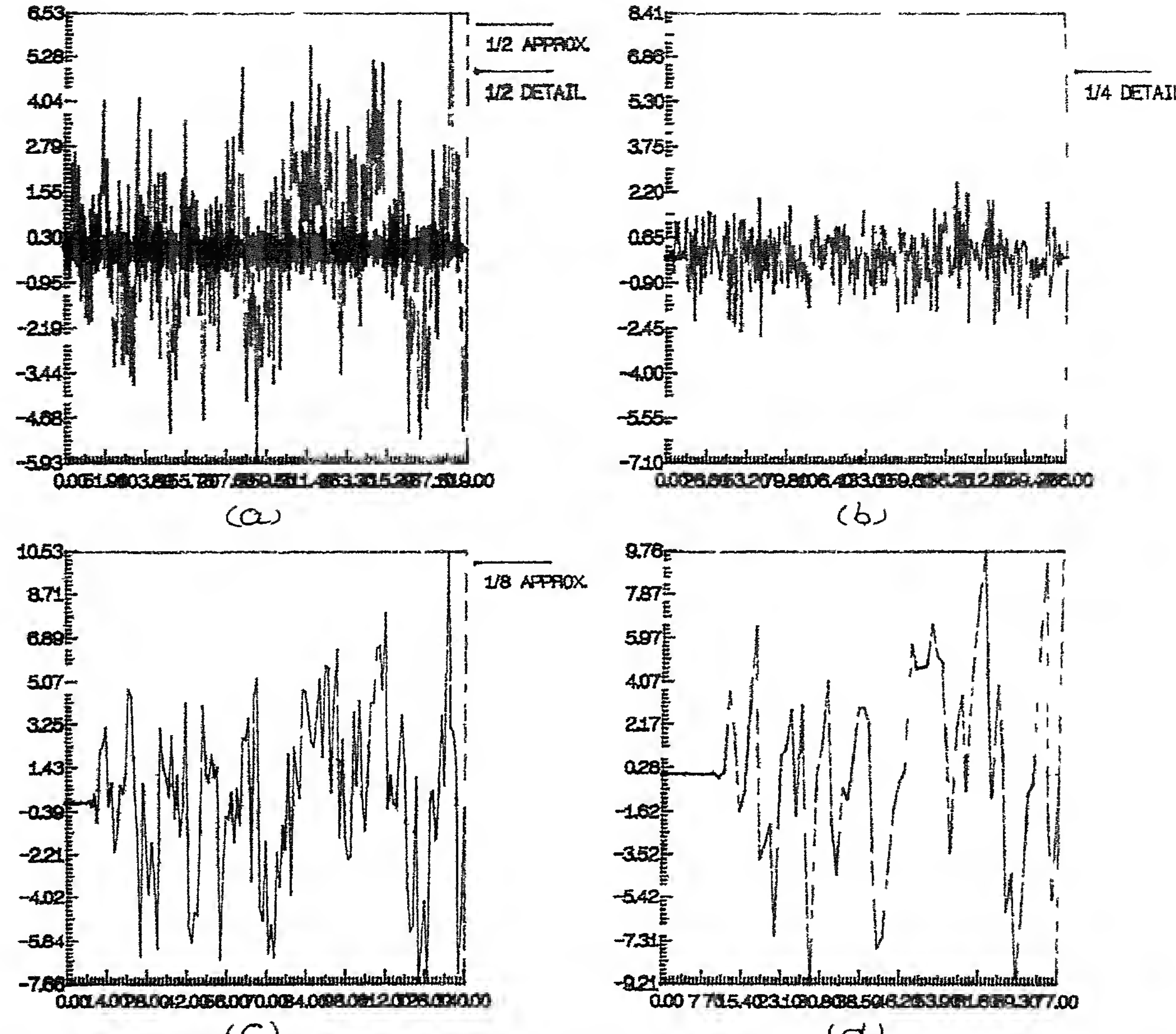


WHITE NOISE DECOMPOSITION

UsPlot

* ngpai * cc-iit/k tejas * 03/13/95 09 58.09 *

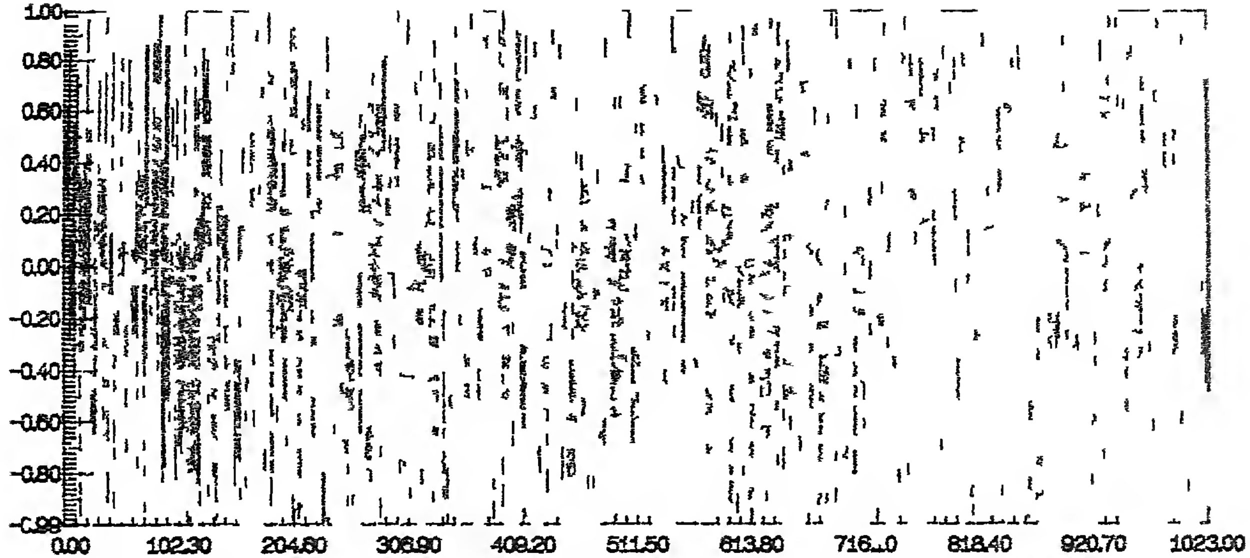
38



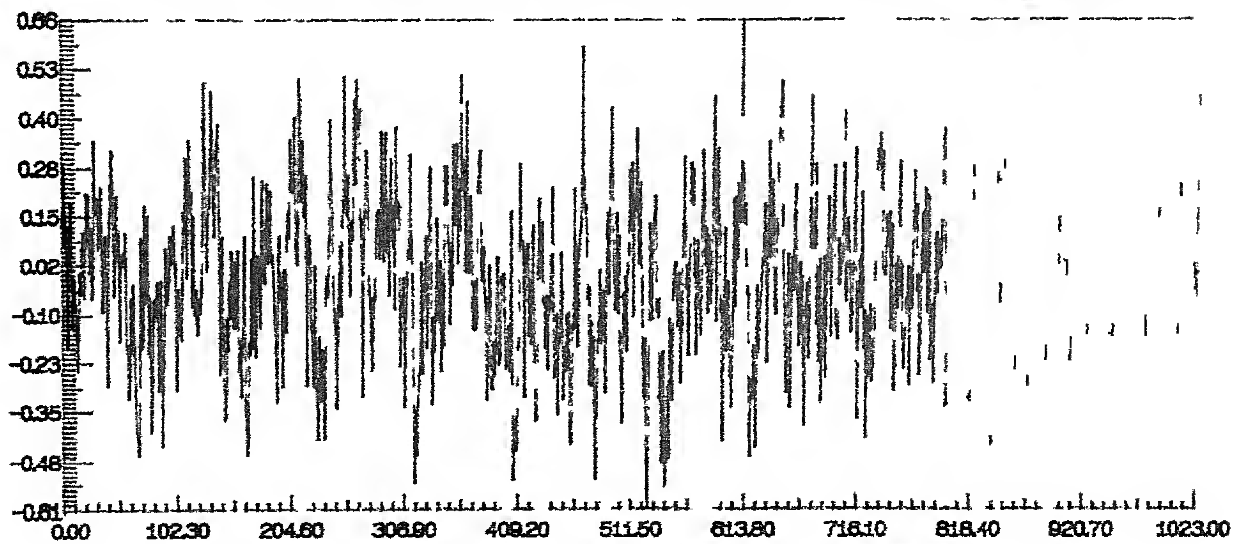
COLOUR NOISE DECOMPOSITION

UsPlot

* ngpai * cc-iit/k   tejas * 03/13/95 10 40 54 *



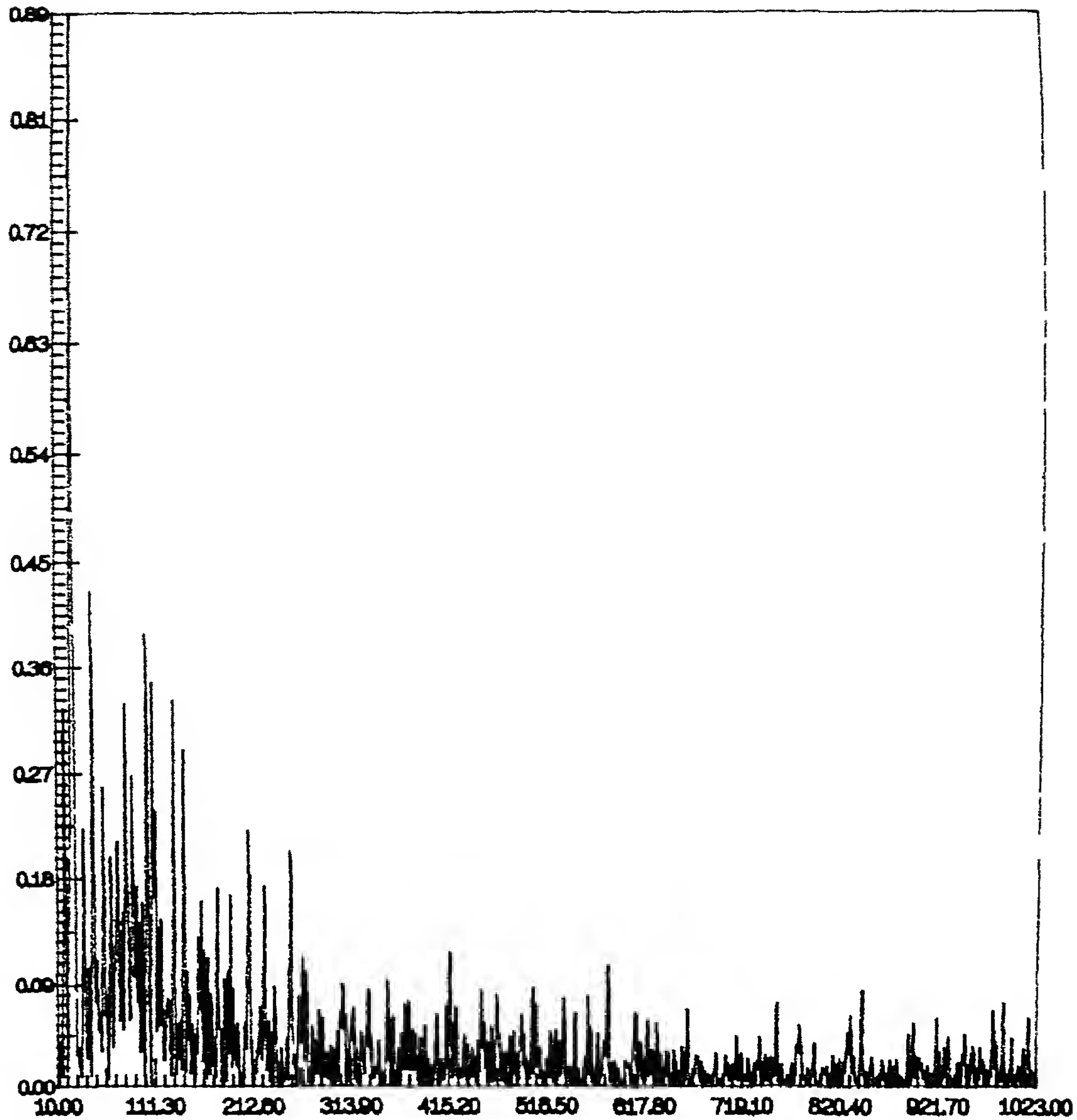
WHITE NOISE SEQUENCE(S) USED IN FILTER-1



1/f NOISE OBTAINED FROM FILTER-1

UsPlot

* ngpai * cc-iit/k tejas * 03/13/95 10:59:23 *



PSD OF 1/f NOISE OBTAINED THROUGH FILTER-1

UsPlot

* asax * cc-iit/k prithvi * 03/13/95 09:49:57 *

Chapter 6

Conclusion

In the present work measurement of low frequency noise through "constant Q" filtering is done All cases of low frequency noise are studied

Simulation of low frequency noise has been done by two methods (1) Matched Z transform technique for analog to digital conversion with analog filter having slope of 10 db/decade, and (2) Wavelet transform based technique It has been found that these techniques give $1/\mid f \mid^{\gamma}$ noise when white noise is applied to the input

It has been found that time for measurement of noise through "Constant Q" filter is more than that required in FFT method in which first autocorrelation of sequence and then it's FFT is taken to get P S D of sequence However with this method modelling of 1/f noise is possible This is the greatest advantage of this method

Appendix A

We know that decomposition and reconstruction is achieved through filters $h(n)$ and $g(n)$ where

$$h(n) = < \phi_{10}, \phi_{0n} > \tag{A 1}$$

$$g(n) = < \psi_{10}, \phi_{0n} > \tag{A 2}$$

Thus to construct $h(n)$ and $g(n)$ we have to start with ϕ satisfying 2 1, 2 2, 2 3, 2 4, 2 5, and then we get $h(n)$ and $g(n)$ from A 1 and A 2 respectively However we can get $h(n)$ and $g(n)$ directly without bothering about ϕ This is due to following theorem [3]

THEOREM A 1 *Let there be a sequence such that,*

1 $\sum_n | h(n) | | n |^{\epsilon} < \infty$ *for some* $\epsilon > 0$

2 $\sum_n h(n-2k)h(n-2l) = \delta_{kl}$

3 $\sum_n h(n) = 2^{1/2}$

suppose also that $m_0(\varepsilon) = 2^{-1/2}\sum_n h(n)e^{jn\epsilon}$

can be written as $m_0(\varepsilon) = [\frac{1}{2}(1+e^{j\epsilon})]^N[\sum_n f(n)e^{jn\epsilon}]$

where

4 $\sum_n | f(n) | | n |^{\epsilon} < \infty$ *for some* $\epsilon > 0$

5 $Sup_{\epsilon \in R} | \sum_n f(n)e^{jn\epsilon} | < 2^{N-1}$

Define

$$g(n) = (-1)^n h(-n+1) \quad \text{(A 3)}$$

$$\Phi(\varepsilon) = (2\pi)^{-1/2} \prod_{j=1}^{\propto} m_0(2^{-j}\varepsilon) \quad \text{(A 4)}$$

$$\psi(x) = 2^{1/2} \sum_n g(n)\phi(2x-n) \quad \text{(A 5)}$$

Where $\Phi(\varepsilon)$ *is fourier transform of* $\phi(x)$

Then $\phi_{jk}(x) = 2^{-j/2}\phi(2^{-j}x - n)$ *define a multiresolution analysis,the* ϕ_{jk} *are the associated orthonormal wavelet basis*

Hence we can choose $h(n)$ directly satisfying first three properties of above theorem, and can apply multiresolution analysis on any signal

A 1 Regularity

A function f(x) is R^{th} order regular if its fourier transform $F(\omega)$ decays according to

$$F(\omega) \sim \Re(|\,\omega\,|^{-R}), \;\; \omega \rightarrow \propto$$

Where the notation $\Re(\;)$ is to be understood to mean that

$lim_{\omega \to \propto} \frac{F(\omega)}{|\omega|^{-R}} < \propto$

Bibliography

[1] KESHNER M S , "$1/f$ noise", Proc IEEE,Vol 70 pp 212 218 March 1982

[2] WORNELL G W ,"Wavelet based representation for the $1/f$ family of fractal processes", proceeding of the IEEE, Vol 81, N0 10, Oct 1993, pp 1428

[3] DAUBECHIES I ,'Communication on pure and applied mathematics" Vol XI I, 909 996 (1988)

[4] MALLAT S G ,"A theory for multiresolution signal decomposition, The wavelet representation' , IEEE transaction on pattern analysis and machine intelligence Vol II No 7, July 1989

[5] DAUBECHIES I ,"The wavelet transform,time frequancy localization and signal analysis " IEEE transtacion on Inf theory Vol 36, No 5, Sept 1990

[6] VETTERLI M 'Wavelets and filter banks theory and design IEEE transaction on signal processing, Vol 40, No 9, Sept 1992, pp 2207

[7] STRANG G ,"Wavelets and dilation equations A brief introduction Society for industrial and applied Mathematics vol 31, No 4, pp 614 627 Dec 1989

[8] RIOUL O ,VETTERLI M , 'Wavelets and signal processing",IEEE SP magazine Oct 1991, pp14

[9] MURCH A R ,BATES R H T ,"Coloured noise generation through deterministic chaos", IEEE trans on circuits and systems 1990, Vol 37, No 5, pp 608

[10] MURCH A R,BATES R H T ,"Deterministic chaotic variably coloured noise Electron letters 1989, Vol 23, No 2

[11] KNUTH D E ,'Seminumerical algorithms, the art of computer programming" Vol 2, reading mass, Addision wesley pub co , pp25 31, 1968

[12] BANDET J S PIERSOL A G ,"Random data, analysis and measurement procedures", 2^{nd} edition

[13] OPPENHEIM A V ,SCHAFER R W ,"Discrete time signal processing", Prentice hall of India pvt ltd, 1992

[14] ANTANIOU A "Digital filters analysis and design', Newyork, McgrawHill 1979

[15] PAPOULIS A ,"Probability, random variables and stochastic processes", 2^{nd} edition 1984